

The Centre of the Schur Algebra

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ABSTRACT. We describe bases of the Schur algebra defined by Schur and Méndez, and explain the relationship between them. In terms of these bases, we describe its structure constants, its centre and its primitive central idempotents.

1. Introduction

Schur introduced the Schur algebra in his doctoral dissertation [7] (a nice exposition can be found in Green's book [3]). He described a basis of this algebra and the structure constants. Méndez [6] gave a graph-theoretic description of Schur's basis, and computed the structure constants in terms of this basis.

In this brief note, we recall these constructions. Using Schur-Weyl duality, we compute the centre of the Schur algebra by interpreting the centre of the symmetric group in it. Using this, we are also able to compute all the primitive central idempotents of the Schur algebra.

2. A basis for the Schur algebra

Let $V = K^n$. Let e_1, \dots, e_n denote the coordinate vectors in V . Then $V^{\otimes d}$ has basis given by vectors of the form

$$e_{\mathbf{i}} = e_{i_1} \otimes \cdots \otimes e_{i_d},$$

where $\mathbf{i} = (i_1, \dots, i_d)$ runs over the Cartesian power $I(n, d) := \{1, \dots, n\}^d$. We may think of $V^{\otimes d}$ as a representation of S_d by the action:

$$\rho(w)(e_{i_1} \otimes \cdots \otimes e_{i_d}) = e_{i_{w(1)}} \otimes \cdots \otimes e_{i_{w(d)}}.$$

The simplest way to define the Schur algebra is the following:

2010 *Mathematics Subject Classification.* 20G43, 20C30, 05E10.

Key words and phrases. Schur algebra, central idempotents, centre.

DEFINITION 1 (Schur Algebra). The Schur algebra $S_K(n, d)$ is defined as

$$S_K(n, d) = \text{End}_{S_d} V^{\otimes d}$$

From this point of view, one of the main results of Schur's dissertation is that the category of representations of $GL_n(K)$ whose matrix coefficients are homogeneous polynomials of degree d is isomorphic to the category of $S_K(n, d)$ -modules (under the assumption that K is infinite).

The symmetric group S_d acts on the set $I(n, d)$ by permuting the d components. Let $K[I(n, d)]$ denote the space of all K -valued functions on $I(n, d)$. This space can be regarded as a representation of S_d using the above action:

$$w \cdot f(i_1, \dots, i_d) = f(i_{w^{-1}(1)}, \dots, i_{w^{-1}(d)})$$

for $w \in S_d$ and $(i_1, \dots, i_d) \in I(n, d)$. Taking the indicator function of (i_1, \dots, i_d) to $e_{i_1} \otimes \dots \otimes e_{i_d}$ gives an isomorphism of $K[I(n, d)]$ onto $V^{\otimes d}$. Thus, the Schur algebra may be viewed as the endomorphism algebra of a permutation representation of S_d .

The endomorphism algebra of a permutation representation is not hard to describe. Suppose, more generally, that G is a finite group, and X is a finite set with a G -action. The space $K[X]$ of K -valued functions on X becomes a representation of G by setting $(g \cdot f)(x) = f(g^{-1}x)$. For any function $k : X \times X \rightarrow K$, define the operator $T_k : K[X] \rightarrow K[X]$ by

$$T_k f(x) = \sum_{y \in X} k(x, y) f(y) \text{ for each } f \in K[X].$$

Let $K[G \backslash (X \times X)]$ denote the space of all K -valued functions on $X \times X$ which are invariant under the diagonal action of G :

$$k(g \cdot x, g \cdot y) = k(x, y) \text{ for all } g \in G, x, y \in X.$$

Then we have:

LEMMA 2. *Let X be a finite G -set. Then the endomorphism algebra of the corresponding permutation representation $K[X]$ is given by*

$$\text{End}_G K[X] = \{T_k \mid k \in K[G \backslash X \times X]\}.$$

In particular, the endomorphism algebra $\text{End}_{S_d} K[I(n, d)]$ has basis indexed by S_d -orbits in $I(n, d)^2$, which we now proceed to describe. Given $\mathbf{i} = (i_1, \dots, i_d)$, and $\mathbf{j} = (j_1, \dots, j_d)$ in $I(n, d)$, form the two-line array:

$$(3) \quad \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix} = \begin{pmatrix} i_1 & i_2 & \dots & i_d \\ j_1 & j_2 & \dots & j_d \end{pmatrix}.$$

There is a unique permutation of the columns of this array which is a generalized permutation in the sense of Knuth [5]:

DEFINITION 4. A generalized permutation is a two line array of the form (3), where the pairs (i_k, j_k) are arranged in non-decreasing lexicographic order from left to right. In other words, the i 's are arranged in increasing order, and the j 's corresponding to the same i are in increasing order.

Generalized permutations with entries in $\{1, \dots, n\}$, in turn correspond to $n \times n$ matrices of non-negative integers whose entries sum to d : the (i, j) th entry of the matrix corresponding to the two line array (3) is given by

$$(5) \quad D(\mathbf{i}, \mathbf{j})_{ij} = \#\{k \mid (i_k, j_k) = (i, j)\}.$$

Thus, the S_d -orbits in $I(n, d)^2$ are in bijective correspondence with the set M_d of non-negative integer $n \times n$ matrices whose entries sum to d .

For each $D \in M_d$, define

$$(6) \quad \xi_D e_{\mathbf{j}} = \sum_{D(\mathbf{i}, \mathbf{j})=D} e_{\mathbf{i}}.$$

Lemma 2, when applied to $X = I(n, d)$ gives us the following basis for $S_K(n, d)$ (see Méndez [6]):

THEOREM 7. *Let $M(n, d)$ denote the set of all $n \times n$ matrices with non-negative integer entries which sum to d . Then*

$$\{\xi_D \mid D \in M(n, d)\}$$

is a basis of $S_K(n, d)$.

In his doctoral dissertation, Schur [7] used the same basis, in a slightly different notation. His basis element represented by

$$\begin{bmatrix} i_1 & \cdots & i_d \\ j_1 & \cdots & j_d \end{bmatrix}$$

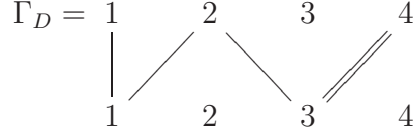
is the same as our basis element ξ_D when D is the integer matrix $D(\mathbf{i}, \mathbf{j})$.

An integer matrix $D = (d_{ij})$ (and thus the corresponding basis element ξ_D of the Schur algebra) may be represented by a bipartite multigraph Γ_D with $n+n$ vertices arranged in two rows with n vertices in each row, and with d edges.

The j th vertex in the first row is connected to the i th vertex in the second row by d_{ij} edges. For example, the matrix

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

corresponds to the multigraph:



We will call the vertex (numbered from 1 to n) in the first row the source, and the vertex in the second row the destination of an edge.

3. Structure Constants

Méndez [6] described the structure constants for the Schur algebra in terms of the bipartite multigraphs described above.

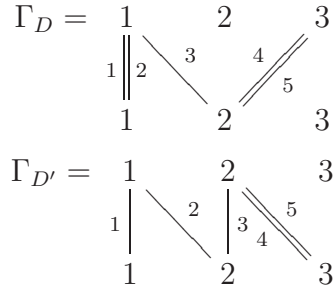
Given two integer multigraphs Γ_D and $\Gamma_{D'}$, consider a bijection

$$f : \text{Edges of } \Gamma_D \rightarrow \text{Edges of } \Gamma_{D'}$$

such that if E is an edge from Γ_D then $f(E)$ is an edge from $\Gamma_{D'}$ whose source is the destination of E . We will consider two such functions equivalent if one is obtained from the other by permuting the edges between the same pair of vertices in either graph. Let $\text{Eul}(D, D')$ denote the set of such functions (this set is often empty) taken up to equivalence.

For example, consider $D = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ and $D' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$. The

corresponding graphs are:



An example of $f \in \text{Eul}(D, D')$ is given by

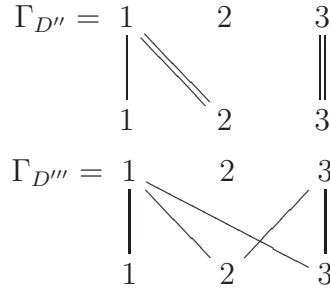
$$f(1) = 1, f(2) = 2, f(3) = 4, f(4) = 3, f(5) = 5.$$

Writing $f(2) = 1$ and $f(1) = 2$ instead of $f(1) = 1$ and $f(2) = 2$ in the above Euler function gives an equivalent Euler function, since the edges 1 and 2 are considered indistinguishable. A complete set of Euler functions (in which f is denoted by the string $f(1)f(2) \cdots f(5)$) is given by

$$\text{Eul}(D, D') = \{12345, 12435\}.$$

For each $f \in \text{Eul}(D, D')$ define the product graph $\Gamma_{DfD'}$ to be the bipartite multigraph where the number of edges from j (in the first row) to i (in the second row) is the number of paths of the form $E, f(E)$ from j to i .

In the running example, the graphs $\Gamma_{DfD'}$ corresponding to the two Euler functions are:



whence

$$D'' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } D''' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

THEOREM 8 (Méndez [6]).

$$\xi_D \xi_{D'} = \sum_{f \in \text{Eul}(D, D')} \xi_{DfD'}.$$

Given three integer matrices D , D' and D'' with sum d , the following result gives an interpretation for the coefficient of ξ_D in $\xi_{D'} \xi_{D''}$:

THEOREM 9. *Let R be the set of all functions r from the set of edges of Γ_D to $\{1, \dots, n\}$ such that*

- (9.1) *The number of edges in D' of type (i, k) is equal to the number of edges e of type $(i, *)$ in D with $r(e) = k$.*
- (9.2) *The number of edges in D'' of type (k, j) is equal to the number of edges e of type $(*, j)$ in D with $r(e) = k$.*

Then the coefficient of ξ_D in $\xi_{D'} \xi_{D''}$ is the cardinality of R .

In the running example, we get

$$\xi_D \xi_{D'} = \xi_{D''} + \xi_{D'''}$$

4. The centre of the Schur algebra

Since $S_K(n, d)$ and $K[S_d]$ are mutual centralizers in $\text{End}_K V^{\otimes d}$, they share a common centre. The centre of $K[S_d]$ consists of class functions. For a permutation $w \in S_d$ let $\rho(w)$ be the partition of d whose parts are the sizes of the cycles in the decomposition of w into disjoint cycles. Therefore, the centre of $K[S_d]$ has a basis indexed by partitions:

$$c_\lambda = \sum_{\rho(w)=\lambda} w.$$

Each c_λ is also an element of the Schur algebra, and therefore admits an expansion:

$$c_\lambda = \sum_D c_{\lambda,D} \xi_D.$$

The following theorem gives the coefficients $c_{\lambda,D}$ for all λ and D :

THEOREM 10. *Suppose the integer matrix D corresponds to the generalized permutation $\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix}$ under the correspondence defined by (5). Then*

$$(11) \quad c_{\lambda,D} = \#\{w \mid \rho(w) = \lambda, w \cdot \mathbf{j} = \mathbf{i}\}$$

PROOF. We have

$$\begin{aligned} \sum_{\rho(w)=\lambda} \rho(w) e_{\mathbf{j}} &= \sum_D c_{\lambda,D} \xi_D e_{\mathbf{j}} \\ &= \sum_D c_{\lambda,D} \sum_{\{\mathbf{i} \mid D(\mathbf{i}, \mathbf{j})=D\}} e_{\mathbf{i}} \quad [\text{by (6)}] \end{aligned}$$

The coefficient of $e_{\mathbf{i}}$ on the right hand side is $c_{\lambda,D}$. On the left hand side it is the number of $w \in S_d$ such that $\rho(w) = \lambda$ and $w \cdot \mathbf{j} = \mathbf{i}$, thereby proving the result. \square

The condition $w \cdot \mathbf{j} = \mathbf{i}$ in (11) implies that if $c_{\lambda,D} \neq 0$ for any λ , then the sum of each row of D is equal to the sum of the corresponding column.

THEOREM 12. *For each partition λ , define*

$$Z_\lambda = \sum_{D \in M_d} c_{\lambda,D} \xi_D,$$

where $c_{\lambda,D}$ is as in (11). Then a basis for the centre of the Schur algebra $S_K(n, d)$ is given by

$$\{Z_\lambda \mid \lambda \text{ is a partition of } d\}.$$

EXAMPLE 13. For $n = 2$ and $d = 4$, we have

$$\begin{aligned} Z_{(4)} &= 6\xi\left(\begin{smallmatrix} 4 & 0 \\ 0 & 0 \end{smallmatrix}\right) + 2\xi\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right) + \xi\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) + 2\xi\left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right) + 2\xi\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2 \end{smallmatrix}\right) + 6\xi\left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right) \\ Z_{(3,1)} &= 8\xi\left(\begin{smallmatrix} 4 & 0 \\ 0 & 0 \end{smallmatrix}\right) + 2\xi\left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right) + 2\xi\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right) + 2\xi\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) + 2\xi\left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right) + 2\xi\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2 \end{smallmatrix}\right) \\ &\quad + 8\xi\left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right) \\ Z_{(2,2)} &= 3\xi\left(\begin{smallmatrix} 4 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \xi\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right) + \xi\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) + 2\xi\left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right) + \xi\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2 \end{smallmatrix}\right) + 3\xi\left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right) \\ Z_{(2,1,1)} &= 6\xi\left(\begin{smallmatrix} 4 & 0 \\ 0 & 0 \end{smallmatrix}\right) + 3\xi\left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \xi\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right) + 2\xi\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) + \xi\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) + 3\xi\left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right) \\ &\quad + \xi\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2 \end{smallmatrix}\right) + 6\xi\left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right) \\ Z_{(1^4)} &= \xi\left(\begin{smallmatrix} 4 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \xi\left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \xi\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) + \xi\left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right) + \xi\left(\begin{smallmatrix} 0 & 0 \\ 0 & 4 \end{smallmatrix}\right) \end{aligned}$$

It is well-known that the primitive central idempotents of semisimple group algebra can be written in terms of the character table (see, for example, Curtis and Reiner [1, Theorem 33.8]). The character values of symmetric groups were computed by Frobenius [2] in 1900 using the theory of symmetric functions and are now very well understood (see James and Kerber [4]). This makes it possible for us to compute the primitive central idempotents of the Schur algebra (this can easily be implemented with a computer):

THEOREM 14. *The primitive central idempotents in the Schur algebra are given by*

$$\epsilon_\lambda = \frac{f_\lambda}{d!} \sum_{\mu \vdash d} \chi_\lambda(\mu) Z_\mu,$$

as λ runs over the set of all partitions of d with at most n parts, where f_λ denotes the number of standard Young tableaux of shape λ , and $\chi_\lambda(\mu)$ is the value of the characters of S_d indexed by the partition λ at the conjugacy class indexed by partition μ .

Acknowledgments

We thank Pooja Singla for her lectures on Schur-Weyl duality, which got us interested in this subject.

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